

A new family of analytic functions defined by means of Rodrigues type formula

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Abstract

In this article, a class of analytic functions is investigated and their some properties are established. Several recurrence relations and various classes of bilinear and bilateral generating functions for these analytic functions are also derived. Examples of some members belonging to this family of analytic functions are given and differential equations satisfied by these functions are also obtained.

Key words and Phrases: Rodrigues formula; Recurrence relation; Generating function; Bilateral and Bilinear generating function; Differential equation; Hermite polynomial.

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1 Introduction

During the recent years, generalized and multivariable forms of the special functions have important role in many branches of mathematics and mathematical physics. Especially, special functions of mathematical physics and their generalizations are often seen in physical problems.

For instance, Hermite polynomials described by Rodrigues formula below [16]

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \quad (1)$$

$$(n = 0, 1, 2, \dots)$$

are generated by

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2).$$

There are several applications of them in mathematics and physics. In mathematics, they are met in probability, such as the Edgeworth series; in combinatorics as an example of an Appell sequence and in the umbral calculus; in physics, they arise in solution of the Schrödinger equation for the harmonic oscillator.

The most remarkable property of the Hermite polynomials is the fact that they are orthogonal polynomials over the interval $(-\infty, \infty)$ with respect to the weight function e^{-x^2} [16]. That is,

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 0 \quad , \quad n \neq m,$$

which is important for their use in quantum mechanics.

In [3], authors, being inspired by the Rodrigues formula for Hermite polynomials, defined the family of polynomials $\phi_{k+n(m-1)}(x)$ via Rodrigues type formula

$$\phi_{k+n(m-1)}(x) = e^{\varphi_m(x)} \frac{d^n}{dx^n} \left(\psi_k(x) e^{-\varphi_m(x)} \right) \quad (2)$$

$$(n = 0, 1, 2, \dots)$$

generated by

$$\sum_{n=0}^{\infty} \frac{\phi_{k+n(m-1)}(x)}{n!} t^n = \psi_k(x+t) e^{\varphi_m(x) - \varphi_m(x+t)} \quad (3)$$

where $\phi_{k+n(m-1)}(x)$ is a polynomial of degree $k + n(m-1)$ since $\varphi_m(x)$ and $\psi_k(x)$ are polynomials of degree m and k , respectively.

Similar to (2), a class of polynomials with two variables defined by Rodrigues type formula and their some properties were investigated in [4]. Moreover, a class of multivariable polynomials was studied in [11].

Now, let's consider a family of analytic functions which is more general than the polynomials (2). Assume that $\varphi_1(x)$, $\varphi_2(x)$ and $\psi(x)$ are analytic functions. Let a family of analytic functions be defined by Rodrigues type formula

$$\Theta_n(x) = \alpha^{\varphi_1(x)} \frac{d^n}{dx^n} \left(\psi(x) \beta^{-\varphi_2(x)} \right) \quad (4)$$

$$(n = 0, 1, 2, \dots)$$

where $\alpha, \beta \in \mathbb{R}^+ \setminus \{1\}$.

It is clear that the special case $\alpha = \beta = e$, $\psi(x) = 1$, $\varphi_1(x) = \varphi_2(x) = x^2$ gives the polynomials $\Theta_n(x) = (-1)^n H_n(x)$ in terms of Hermite polynomials.

To give another special case, we now recall that the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x, y)$ [1] are defined by the generating function

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \exp(xt + yt^2).$$

In the special case $\alpha = \beta = e$, $\psi(x) = 1$, $\varphi_1(x) = \varphi_2(x) = -x^2$, the functions given by (4) reduce to the polynomials $\Theta_n\left(\frac{x}{2}\right) = H_n(x, 1)$.

The main purpose of the present paper is to derive a generating function by means of Cauchy's integral formula and to investigate some properties of these functions. The set up of this paper is summarized as follows. In section 2, we give a generating function satisfied by this family of analytic functions and then, by using this generating function we obtain several recurrence relations. In addition, we find differential equations satisfied by some special families of analytic functions defined by (4), depending on choices of $\psi(x)$ and $\varphi_2(x)$. Section 3 is devoted to prove a theorem to find various families of bilateral and bilinear generating functions and then, to apply this theorem to the special cases.

2 A Generating Function and Recurrence Relations

In literature, there are numerous investigations to obtain generating functions and recurrence relations satisfied by special functions and polynomials (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18]). It is possible to derive a recurrence relation by using a generating function. In this section, we first try to find a generating function for the family of analytic functions $\Theta_n(x)$. Afterwards, we give some recurrence relations with the help of this generating function.

Now, we start with the following theorem.

Theorem 1. *The family of analytic functions given by (4) has the following generating function*

$$\sum_{n=0}^{\infty} \frac{\Theta_n(x)}{n!} t^n = \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)} \quad (5)$$

where $\varphi_1(x)$, $\varphi_2(x)$ and $\psi(x)$ are analytic functions and $\alpha, \beta \in \mathbb{R}^+ \setminus \{1\}$.

Proof. By considering the Cauchy's integral formula

$$\frac{d^n}{dx^n} \left(\psi(x) \beta^{-\varphi_2(x)} \right) = \frac{n!}{2\pi i} \oint_C \frac{\psi(z) \beta^{-\varphi_2(z)} dz}{(z-x)^{n+1}} \quad (6)$$

for a suitable contour C , we can write

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\Theta_n(x)}{n!} t^n &= \alpha^{\varphi_1(x)} \sum_{n=0}^{\infty} \frac{t^n}{2\pi i} \oint_C \frac{\psi(z) \beta^{-\varphi_2(z)} dz}{(z-x)^{n+1}} \\ &= \frac{\alpha^{\varphi_1(x)}}{2\pi i} \oint_C \frac{\psi(z) \beta^{-\varphi_2(z)}}{z-x} \sum_{n=0}^{\infty} \left(\frac{t}{z-x} \right)^n dz \\ &= \frac{\alpha^{\varphi_1(x)}}{2\pi i} \oint_C \frac{\psi(z) \beta^{-\varphi_2(z)}}{z-x} \frac{1}{1 - \frac{t}{z-x}} dz \\ &= \frac{\alpha^{\varphi_1(x)}}{2\pi i} \oint_C \frac{\psi(z) \beta^{-\varphi_2(z)}}{z-(x+t)} dz \end{aligned}$$

for

$$\left| \frac{t}{z-x} \right| < 1.$$

If we take into account Cauchy's integral formula (6) again, we conclude that

$$\sum_{n=0}^{\infty} \frac{\Theta_n(x)}{n!} t^n = \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)}$$

where the point $x+t$ should also be inside the contour C . □

Let's use the above theorem to obtain various recurrence relations for the functions (4). For convenience, let the right side of the generating function (5) be denoted by

$$F(x, t) = \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)}.$$

Theorem 2. *For the family of analytic functions given by (4), we have*

$$\begin{aligned} &\sum_{p=0}^n \binom{n}{p} \left\{ \Theta_{n-p+1}(x) \psi^{(p)}(x) - \Theta_{n-p}(x) \psi^{(p+1)}(x) \right\} \\ &= -\ln \beta \sum_{k=0}^n \sum_{p=0}^{n-k} \binom{n}{k} \binom{n-k}{p} \Theta_{n-p-k}(x) \psi^{(p)}(x) \varphi_2^{(k+1)}(x) \end{aligned} \quad (7)$$

and

$$\begin{aligned}
& \sum_{p=0}^n \binom{n}{p} \left\{ \Theta'_{n-p}(x) \psi^{(p)}(x) - \ln \alpha \varphi'_1(x) \Theta_{n-p}(x) \psi^{(p)}(x) \right. \\
& \quad \left. - \Theta_{n-p}(x) \psi^{(p+1)}(x) \right\} \\
& = -\ln \beta \sum_{k=0}^n \sum_{p=0}^{n-k} \binom{n}{k} \binom{n-k}{p} \Theta_{n-p-k}(x) \psi^{(p)}(x) \varphi_2^{(k+1)}(x)
\end{aligned} \tag{8}$$

for $n \geq 0$.

Proof. If we take the derivative of the function $F(x, t) = \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)}$ with respect to t , we arrive at

$$\psi(x+t) \frac{\partial}{\partial t} F(x, t) = \left\{ -\ln \beta \varphi'_2(x+t) \psi(x+t) + \psi'(x+t) \right\} F(x, t). \tag{9}$$

Considering the left side of the generating function (5) in (9) gives

$$\psi(x+t) \sum_{n=0}^{\infty} \Theta_{n+1}(x) \frac{t^n}{n!} = \left\{ -\ln \beta \varphi'_2(x+t) \psi(x+t) + \psi'(x+t) \right\} \sum_{n=0}^{\infty} \Theta_n(x) \frac{t^n}{n!}.$$

If we use Taylor series of the analytic functions $\psi(x+t)$, $\psi'(x+t)$ and $\varphi'_2(x+t)$ at $t=0$, respectively

$$\begin{aligned}
\psi(x+t) &= \sum_{p=0}^{\infty} \psi^{(p)}(x) \frac{t^p}{p!}, \\
\psi'(x+t) &= \sum_{p=0}^{\infty} \psi^{(p+1)}(x) \frac{t^p}{p!}
\end{aligned}$$

and

$$\varphi'_2(x+t) = \sum_{k=0}^{\infty} \varphi_2^{(k+1)}(x) \frac{t^k}{k!},$$

we have

$$\begin{aligned}
\sum_{n,p=0}^{\infty} \Theta_{n+1}(x) \psi^{(p)}(x) \frac{t^{n+p}}{n!p!} &= -\ln \beta \sum_{n,k,p=0}^{\infty} \Theta_n(x) \varphi_2^{(k+1)}(x) \psi^{(p)}(x) \frac{t^{n+k+p}}{n!k!p!} \\
&\quad + \sum_{n,p=0}^{\infty} \Theta_n(x) \psi^{(p+1)}(x) \frac{t^{n+p}}{n!p!}.
\end{aligned}$$

Upon inverting the order of summation above, if we replace n by $n-p$, we can write

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{p=0}^n \Theta_{n-p+1}(x) \psi^{(p)}(x) \frac{t^n}{(n-p)!p!} \\
& = -\ln \beta \sum_{n,k=0}^{\infty} \sum_{p=0}^n \Theta_{n-p}(x) \varphi_2^{(k+1)}(x) \psi^{(p)}(x) \frac{t^{n+k}}{(n-p)!k!p!} \\
& \quad + \sum_{n=0}^{\infty} \sum_{p=0}^n \Theta_{n-p}(x) \psi^{(p+1)}(x) \frac{t^n}{(n-p)!p!}.
\end{aligned}$$

If we take $n-k$ instead of n in the first summation in the right side of this equation and then compare the coefficients of $\frac{t^n}{n!}$, we complete the proof of (7).

On the other hand, it is easily seen that $F(x, t)$ satisfies

$$\psi(x+t) \frac{\partial}{\partial x} F(x, t) = \left\{ \ln \alpha \varphi_1'(x) \psi(x+t) - \ln \beta \varphi_2'(x+t) \psi(x+t) + \psi'(x+t) \right\} F(x, t).$$

In order to obtain (8), it is enough to make similar calculations above. □

Corollary 1. *Combining the recurrence relations in Theorem 2 gives*

$$\begin{aligned} & \sum_{p=0}^n \left\{ \Theta_{n-p+1}(x) \psi^{(p)}(x) - \Theta_{n-p}(x) \psi^{(p+1)}(x) - \Theta'_{n-p}(x) \psi^{(p)}(x) \right. \\ & \left. + \ln \alpha \varphi_1'(x) \Theta_{n-p}(x) \psi^{(p)}(x) + \Theta_{n-p}(x) \psi^{(p+1)}(x) \right\} \binom{n}{p} = 0 \end{aligned}$$

for $n \geq 0$.

Theorem 3. *The family of analytic functions given by (4) satisfies*

$$\Theta'_n(x) = \Theta_{n+1}(x) + \ln \alpha \varphi_1'(x) \Theta_n(x)$$

for $n \geq 0$.

Proof. For the function $F(x, t) = \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)}$, the following equation holds

$$\frac{\partial}{\partial x} F(x, t) = \frac{\partial}{\partial t} F(x, t) + \ln \alpha \varphi_1'(x) F(x, t). \quad (10)$$

If we take into consideration generating function (5) in (10), we can write

$$\sum_{n=0}^{\infty} \Theta'_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \Theta_{n+1}(x) \frac{t^n}{n!} + \ln \alpha \varphi_1'(x) \sum_{n=0}^{\infty} \Theta_n(x) \frac{t^n}{n!}.$$

By equating the coefficients of $\frac{t^n}{n!}$, it follows that

$$\Theta'_n(x) = \Theta_{n+1}(x) + \ln \alpha \varphi_1'(x) \Theta_n(x)$$

for $n \geq 0$. □

Theorem 4. *An another recurrence relation for the analytic functions $\Theta_n(x)$ is as follows*

$$\begin{aligned} & \sum_{p=0}^n \left\{ \ln \alpha \Theta_{n-p+1}(x) \psi^{(p)}(x) \varphi_1'(x) + \Theta_{n-p+1}(x) \psi^{(p+1)}(x) \right. \\ & \left. - \Theta'_{n-p}(x) \psi^{(p+1)}(x) \right\} \binom{n}{p} \\ & + \ln \beta \sum_{k=0}^n \sum_{p=0}^{n-k} \left\{ \binom{n}{k} \binom{n-k}{p} \Theta'_{n-k-p}(x) \varphi_2^{(k+1)}(x) \psi^{(p)}(x) \right. \\ & \left. - \binom{n}{k} \binom{n-k}{p} \Theta_{n-k-p+1}(x) \psi^{(p)}(x) \varphi_2^{(k+1)}(x) \right\} = 0 \end{aligned} \quad (11)$$

for $n \geq 0$.

Proof. It is obvious from the generating function (5) that

$$\begin{aligned} & \{\ln \alpha \varphi_1'(x) \psi(x+t) - \ln \beta \psi(x+t) \varphi_2'(x+t) + \psi'(x+t)\} \frac{\partial}{\partial t} F(x, t) \\ &= \{\psi'(x+t) - \ln \beta \psi(x+t) \varphi_2'(x+t)\} \frac{\partial}{\partial x} F(x, t) \end{aligned}$$

from which, we can obtain the desired relation. \square

Corollary 2. *If we combine Theorem 3 and Theorem 4 for the special case $\psi(x) = 1$, we obtain*

$$\Theta_{n+1}(x) = -\ln \beta \sum_{k=0}^n \binom{n}{k} \varphi_2^{(k+1)}(x) \Theta_{n-k}(x)$$

for $n \geq 0$.

Now, by means of the recurrence relations given above, we show that some special cases of the analytic functions given by (4) are solutions of a differential equation.

Let $\psi(x) = 1$. Then from (4), we have

$$\Theta_n(x) = \alpha^{\varphi_1(x)} \frac{d^n}{dx^n} \beta^{-\varphi_2(x)}, \quad (12)$$

which verifies the generating function

$$\sum_{n=0}^{\infty} \frac{\Theta_n(x)}{n!} t^n = \alpha^{\varphi_1(x)} \beta^{-\varphi_2(x+t)}.$$

Theorem 5. *Let $\varphi_2(x)$ be a polynomial of degree 2 and $\varphi_1(x)$, not constant, be an analytic function. For the functions $y = \Theta_n(x)$ defined by (12), one easily gets*

$$\begin{aligned} & y'' + \{\ln \beta \varphi_2'(x) - 2 \ln \alpha \varphi_1'(x)\} y' + \{\ln^2 \alpha (\varphi_1'(x))^2 - \ln \alpha \varphi_1''(x) \\ & - \ln \alpha \ln \beta \varphi_1'(x) \varphi_2'(x) + (n+1) \ln \beta \varphi_2''(x)\} y = 0. \end{aligned}$$

Proof. To prove this theorem, it is enough to combine Theorem 3 and Theorem 4. \square

Example 1. *In the special case $\varphi_1(x) = \varphi_2(x) = x^2$, $\alpha = \beta = e$, it is obvious that $\Theta_n(x) = (-1)^n H_n(x)$ satisfies Hermite differential equation*

$$y'' - 2xy' + 2ny = 0.$$

Theorem 6. *Assume that $\varphi_2(x)$ is a polynomial of degree 3 and $\varphi_1(x)$ is an analytic function which is not constant. Then the family of functions $y = \Theta_n(x)$ satisfies the third order linear differential equation*

$$\begin{aligned} & y''' + \{\ln \beta \varphi_2'(x) - 3 \ln \alpha \varphi_1'(x)\} y'' + \{(n+2) \ln \beta \varphi_2''(x) - 3 \ln \alpha \varphi_1''(x) \\ & - 2 \ln \alpha \ln \beta \varphi_1'(x) \varphi_2'(x) + 3 \ln^2 \alpha (\varphi_1'(x))^2\} y' + \{-\ln^3 \alpha (\varphi_1'(x))^3 \\ & + \ln^2 \alpha \ln \beta \varphi_2'(x) (\varphi_1'(x))^2 + 3 \ln^2 \alpha \varphi_1'(x) \varphi_1''(x) - \ln \alpha \varphi_1'''(x) \\ & - \ln \alpha \ln \beta \varphi_1''(x) \varphi_2'(x) - (n+2) \ln \alpha \ln \beta \varphi_2''(x) \varphi_1'(x) \\ & + \frac{1}{2} (n+1) (n+2) \ln \beta \varphi_2'''(x)\} y = 0. \end{aligned}$$

Proof. As $\psi(x) = 1$ and $\varphi_2(x)$ is a polynomial of degree 3, then the relation given by (11) reduces to

$$\begin{aligned} & \ln \alpha \Theta_{n+1}(x) \varphi_1'(x) + \ln \beta \{ \varphi_2'(x) (\Theta_n'(x) - \Theta_{n+1}(x)) \\ & + \varphi_2''(x) (\Theta_{n-1}'(x) - \Theta_n(x)) n \\ & + \frac{1}{2} n(n-1) \varphi_2'''(x) (\Theta_{n-2}'(x) - \Theta_{n-1}(x)) \} = 0 \end{aligned}$$

Replacing n by $n+2$, it follows that

$$\begin{aligned} & \ln \alpha \Theta_{n+3}(x) \varphi_1'(x) + \ln \beta \{ \varphi_2'(x) (\Theta_{n+2}'(x) - \Theta_{n+3}(x)) \\ & + \varphi_2''(x) (\Theta_{n+1}'(x) - \Theta_{n+2}(x)) (n+2) \\ & + \frac{1}{2} (n+1)(n+2) \varphi_2'''(x) (\Theta_n'(x) - \Theta_{n+1}(x)) \} = 0. \end{aligned} \tag{13}$$

On the other hand, we have the following recurrence relation from Theorem 3

$$\Theta_{n+1}(x) = \Theta_n'(x) - \ln \alpha \varphi_1'(x) \Theta_n(x),$$

from which, by repeating this recurrence relation, we can obtain

$$\begin{aligned} \Theta_{n+1}'(x) &= \Theta_n''(x) - \ln \alpha \varphi_1'(x) \Theta_n'(x) - \ln \alpha \varphi_1''(x) \Theta_n(x), \\ \Theta_{n+2}(x) &= \Theta_n''(x) - 2 \ln \alpha \varphi_1'(x) \Theta_n'(x) + \left\{ \ln^2 \alpha (\varphi_1'(x))^2 - \ln \alpha \varphi_1''(x) \right\} \Theta_n(x), \\ \Theta_{n+2}'(x) &= \Theta_n'''(x) - 2 \ln \alpha \varphi_1'(x) \Theta_n''(x) + \left\{ \ln^2 \alpha (\varphi_1'(x))^2 - 3 \ln \alpha \varphi_1''(x) \right\} \Theta_n'(x) \\ &\quad + \left\{ 2 \ln^2 \alpha \varphi_1'(x) \varphi_1''(x) - \ln \alpha \varphi_1'''(x) \right\} \Theta_n(x), \\ \Theta_{n+3}(x) &= \Theta_n'''(x) - 3 \ln \alpha \varphi_1'(x) \Theta_n''(x) + \left\{ 3 \ln^2 \alpha (\varphi_1'(x))^2 - 3 \ln \alpha \varphi_1''(x) \right\} \Theta_n'(x) \\ &\quad + \left\{ -\ln^3 \alpha (\varphi_1'(x))^3 + 3 \ln^2 \alpha \varphi_1'(x) \varphi_1''(x) - \ln \alpha \varphi_1'''(x) \right\} \Theta_n(x). \end{aligned}$$

Taking into account these equalities in (13) leads to the desired third order linear differential equation. \square

Similarly, as a result of Theorem 3 and Theorem 4, it is possible to give the next theorem.

Theorem 7. *Let $\varphi_2(x)$ be a polynomial of degree 4 and $\varphi_1(x)$, not constant, be an analytic function. The*

functions $y = \Theta_n(x)$ are solutions of the fourth order linear differential equation

$$\begin{aligned}
& y^{(iv)} + \{\ln \beta \varphi_2'(x) - 4 \ln \alpha \varphi_1'(x)\} y''' + \left\{ 6 \ln^2 \alpha (\varphi_1'(x))^2 \right. \\
& - 3 \ln \alpha \ln \beta \varphi_1'(x) \varphi_2'(x) - 6 \ln \alpha \varphi_1''(x) + (n+3) \ln \beta \varphi_2''(x) \} y'' \\
& + \left\{ -4 \ln^3 \alpha (\varphi_1'(x))^3 + 3 \ln^2 \alpha \ln \beta \varphi_2'(x) (\varphi_1'(x))^2 + 12 \ln^2 \alpha \varphi_1'(x) \varphi_1''(x) \right. \\
& - 4 \ln \alpha \varphi_1'''(x) - 3 \ln \alpha \ln \beta \varphi_1''(x) \varphi_2'(x) - 2(n+3) \ln \alpha \ln \beta \varphi_2''(x) \varphi_1'(x) \\
& + \frac{1}{2} (n+2)(n+3) \ln \beta \varphi_2'''(x) \} y' + \left\{ \ln^4 \alpha (\varphi_1'(x))^4 - \ln^3 \alpha \ln \beta (\varphi_1'(x))^3 \varphi_2'(x) \right. \\
& - 6 \ln^3 \alpha (\varphi_1'(x))^2 \varphi_1''(x) + 4 \ln^2 \alpha \varphi_1'(x) \varphi_1'''(x) + 3 \ln^2 \alpha \ln \beta \varphi_1'(x) \varphi_1''(x) \varphi_2'(x) \\
& - \ln \alpha \ln \beta \varphi_1'''(x) \varphi_2'(x) - 3 \ln^2 \alpha (\varphi_1''(x))^2 - \ln \alpha \varphi_1^{(iv)}(x) \\
& + (n+3) \ln^2 \alpha \ln \beta (\varphi_1'(x))^2 \varphi_2''(x) - (n+3) \ln \alpha \ln \beta \varphi_1''(x) \varphi_2''(x) \\
& \left. - \frac{1}{2} (n+2)(n+3) \ln \alpha \ln \beta \varphi_1'(x) \varphi_2'''(x) + \frac{1}{6} (n+1)(n+2)(n+3) \ln \beta \varphi_2^{(iv)}(x) \right\} y = 0
\end{aligned}$$

Example 2. For example, taking $\varphi_1(x) = \varphi_2(x) = -x^4$, $\alpha = \beta = e$ in the equation (12), we have

$$\Theta_n(x) = e^{-x^4} \frac{d^n}{dx^n} (e^{x^4}) \quad , \quad n = 0, 1, 2, \dots,$$

which gives a family of polynomials of degree $3n$. Theorem 7 shows that these polynomials are solutions of the following fourth order linear differential equation

$$\begin{aligned}
& y^{(4)} + 12x^3 y''' + \{48x^6 - 12(n-3)x^2\} y'' \\
& + \{64x^9 + (144 - 96n)x^5 - 12(n^2 + 5n - 2)x\} y' \\
& + \{-192nx^8 - 48(n^2 + 8n)x^4 - 4n(n^2 + 6n + 11)\} y = 0.
\end{aligned}$$

Remark 1. We observe that if $\varphi_2(x)$ is a polynomial of degree m ($m \leq n$) and $\varphi_1(x)$ is an analytic function, not constant, then the functions $y = \Theta_n(x)$ satisfy the m -th order linear differential equation. But, it is complicated to obtain its explicit form.

3 Bilinear and Bilateral Generating Functions

In this section, we try to derive many families of bilinear and bilateral generating functions for the family of analytic functions given by (4) by means of the similar method presented in [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18].

For this purpose, let's begin the following theorem.

Theorem 8. Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, \dots, y_s)$ of s complex variables y_1, \dots, y_s ($s \in \mathbb{N}$) and of complex order μ , if

$$\begin{aligned}
\Lambda_{\mu, \nu}(y_1, \dots, y_s; z) &:= \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, \dots, y_s) z^k \\
&(a_k \neq 0, \quad \mu, \nu \in \mathbb{C})
\end{aligned} \tag{14}$$

and

$$\Phi_{n,p,\mu,\nu}(x; y_1, \dots, y_s; \zeta) := \sum_{k=0}^{[n/p]} \frac{a_k}{(n-pk)!} \Theta_{n-pk}(x) \Omega_{\mu+\nu k}(y_1, \dots, y_s) \zeta^k \quad (15)$$

$(n, p \in \mathbb{N})$

then it follows

$$\sum_{n=0}^{\infty} \Phi_{n,p,\mu,\nu}(x; y_1, \dots, y_s; \frac{\eta}{t^p}) t^n = \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)} \Lambda_{\mu,\nu}(y_1, \dots, y_s; \eta) \quad (16)$$

provided that each member of (16) exists.

Proof. For convenience, let S denote the left side of (16) in Theorem 8. If we substitute the explicit form of polynomials

$$\Phi_{n,p,\mu,\nu}(x; y_1, \dots, y_s; \frac{\eta}{t^p})$$

from the definition (15) in (16), we can write

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} \frac{a_k}{(n-pk)!} \Theta_{n-pk}(x) \Omega_{\mu+\nu k}(y_1, \dots, y_s) \eta^k t^{n-pk}. \quad (17)$$

Upon inverting the order of summation in (17), if we replace n by $n + pk$, it follows that

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_k}{n!} \Theta_n(x) \Omega_{\mu+\nu k}(y_1, \dots, y_s) \eta^k t^n \\ &= \sum_{n=0}^{\infty} \Theta_n(x) \frac{t^n}{n!} \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, \dots, y_s) \eta^k \\ &= \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)} \Lambda_{\mu,\nu}(y_1, \dots, y_s; \eta), \end{aligned}$$

which completes the proof. □

The above theorem can be used to construct various families of bilateral and bilateral generating functions by expressing the multivariable function

$$\Omega_{\mu+\nu k}(y_1, \dots, y_s) \quad (k \in \mathbb{N}_0, \quad s \in \mathbb{N})$$

in terms of simpler function of one and more variables.

For example, taking

$$s = 1 \quad \text{and} \quad \Omega_{\mu+\nu k}(y) = \mathcal{B}_{\mu+\nu k}^{(\alpha)}(y; \lambda),$$

where $\mathcal{B}_n^{(\alpha)}(x; \lambda)$ ($\lambda \in \mathbb{C}$) denotes Apostol-Bernoulli polynomials of order $\alpha \in \mathbb{N}_0$ which are defined by the generating function [15]

$$\left(\frac{t}{\lambda e^t - 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \quad (18)$$

$$(|t| < 2\pi, \text{ when } \lambda = 1 \quad ; \quad |t| < |\log \lambda|, \text{ when } \lambda \neq 1),$$

leads to a class of bilateral generating functions for Apostol-Bernoulli polynomials and the family of analytic functions defined by (4).

Corollary 3. If $\Lambda_{\mu,\nu}(y; z) := \sum_{k=0}^{\infty} a_k \mathcal{B}_{\mu+\nu k}^{(\alpha)}(y; \lambda) z^k$ where $a_k \neq 0$, $\nu, \mu \in \mathbb{C}$; and

$$\begin{aligned} & \Phi_{n,p,\mu,\nu}(x; y; \zeta) \\ &:= \sum_{k=0}^{[n/p]} \frac{a_k}{(n-pk)!} \Theta_{n-pk}(x) \mathcal{B}_{\mu+\nu k}^{(\alpha)}(y; \lambda) \zeta^k \end{aligned}$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu} \left(x; y; \frac{\eta}{t^p} \right) t^n = \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)} \Lambda_{\mu,\nu}(y; \eta) \quad (19)$$

provided that each member of (19) exists.

Remark 2. If we use the generating relation (18) for Apostol-Bernoulli polynomials by taking $a_k = \frac{1}{k!}$, $\mu = 0$, $\nu = 1$, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} \Theta_{n-pk}(x) \mathcal{B}_k^{(\alpha)}(y; \lambda) \frac{\eta^k}{k!} \frac{t^{n-pk}}{(n-pk)!} \\ &= \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)} \left(\frac{\eta}{\lambda e^{\eta} - 1} \right)^{\alpha} e^{y\eta}, \end{aligned}$$

where

$$(|\eta| < 2\pi, \text{ when } \lambda = 1 ; \quad |\eta| < |\log \lambda|, \text{ when } \lambda \neq 1).$$

In a similar manner, choosing $s = 1$ and $\Omega_{\mu+\nu k}(y) = \Theta_{\mu+\nu k}(y)$ in Theorem 8, we obtain the following class of bilinear generating functions for the functions generated by (5).

Corollary 4. If

$$\Lambda_{\mu,\nu}(y; z) := \sum_{k=0}^{\infty} a_k \Theta_{\mu+\nu k}(y) z^k,$$

where $a_k \neq 0$, $\mu, \nu \in \mathbb{C}$ and

$$\begin{aligned} & \Phi_{n,p,\mu,\nu}(x; y; \zeta) \\ &:= \sum_{k=0}^{[n/p]} a_k \Theta_{\mu+\nu k}(y) \frac{\Theta_{n-pk}(x)}{(n-pk)!} \zeta^k \end{aligned}$$

where $n, p \in \mathbb{N}$. Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \Phi_{n,p,\mu,\nu} \left(x; y; \frac{\eta}{t^p} \right) t^n &= \alpha^{\varphi_1(x)} \psi(x+t) \beta^{-\varphi_2(x+t)} \\ &\quad \times \Lambda_{\mu,\nu}(y; \eta) \end{aligned} \quad (20)$$

provided that each member of (20) exists.

Remark 3. Getting $a_k = \frac{1}{k!}$, $\mu = 0$, $\nu = 1$ and then taking into account the generating function (5) give the bilinear generating function for the analytic functions $\Theta_n(x)$

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} \frac{\Theta_k(y)}{k!} \frac{\Theta_{n-pk}(x)}{(n-pk)!} \eta^k t^{n-pk} \\ &= \alpha^{\varphi_1(x)+\varphi_1(y)} \psi(x+t) \psi(y+\eta) \beta^{-\varphi_2(x+t)-\varphi_2(y+\eta)} \end{aligned}$$

Besides, it is possible to get multivariable function $\Omega_{\mu+\psi k}(y_1, \dots, y_s)$, ($s \in \mathbb{N}$) as an appropriate product of several simpler functions for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$). Thus, Theorem 8 can be applied in order to derive various families of multilinear and multilateral generating functions for the functions generated by (5).

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